ΑΣΚΗΣΗ 3✓ (σελ. 113)

$$y''' - 3y'' + 3y' - y = x^2 + 5e^x$$

<u>ΛΥΣΗ</u>

(E_0): $\lambda^3 - 3\lambda^2 + 3\lambda - 1 = 0$

$(\lambda - 1)^3 = 0$

$\lambda = 1$ πολ/τα. 3

ΒΣΛ: $\{e^x, x \cdot e^x, x^2 \cdot e^x\}$

(E_1): $y''' - 3y'' + 3y' - y = x^2$

Θέτω $y_\mu^{(1)}(x) = \alpha x^2 + \beta x + \gamma$

$-3 \cdot 2\alpha + 3(2\alpha x + \beta) - \alpha x^2 - \beta x - \gamma = x^2$

$-6\alpha + 6\alpha x + 3\beta - \alpha x^2 - \beta x - \gamma = x^2$

$-\alpha x^2 + (6\alpha - \beta)x - 6\alpha + 3\beta - \gamma = x^2$

$\alpha = -1$

$6\alpha - \beta = 0 \Rightarrow \beta = 6\alpha = -6$

$\gamma = -6\alpha - 3\beta = 24$

Άρα $y_\mu^{(1)} = -x^2 - 6x - 12$

(E_2): $y''' - 3y'' + 3y' - y = 5e^x$

Θέτω $y = z e^x$

$z''' \cdot e^x + 3z'' \cdot e^x + 3z' \cdot e^x + e^x z - 3(z'' \cdot e^x + 3z' e^x + z e^x) +$

$+ 3(z' e^x + z e^x) - z e^x = 5 e^x \Rightarrow$

$\Rightarrow z''' + 3z'' + 3z' + z - 3z'' - 6z' - 3z + 3z' + 3z - z = 5$

$\Rightarrow z''' = 5 \Rightarrow z(x) = \frac{5}{6} x^3$

$y_\mu^{(2)} = \frac{5}{6} x^3 \cdot e^x$

έτσι $y_\mu(x) = y_\mu^{(1)}(x) + y_\mu^{(2)}(x)$.

ΠΑΡΑΔΕΙΓΜΑ 5

$y'' - 2y' + y = \frac{1}{x} e^x, \ x > 0$

ΛΥΣΗ

ΟΜΟΓΕΝΗΣ (E_0): $y'' - 2y' + y = 0 \overset{\text{χπ}}{\rightsquigarrow} \lambda^2 - 2\lambda + 1 = 0 \Rightarrow$

$\Rightarrow (\lambda - 1)^2 = 0 \Rightarrow \lambda = 1$ ΔΙΠΛΗ

ΒΕΛ $\{e^x, xe^x\}$

$W(x) = \begin{vmatrix} e^x & xe^x \\ e^x & xe^x + e^x \end{vmatrix} = e^{2x}$

$W_1(x) = \begin{vmatrix} 0 & xe^x \\ 1 & xe^x + e^x \end{vmatrix} = -xe^x$ & $W_2(x) = \begin{vmatrix} e^x & 0 \\ e^x & 1 \end{vmatrix} = e^x$

$y_\mu(x) = y_1(x) \int_1^x -\frac{\cancel{s} \cancel{e^s}}{\cancel{e^{2s}}} \cdot \frac{1}{\cancel{s}} e^s \, ds + y_2(x) \int_1^x \frac{\cancel{e^s}}{\cancel{e^{2s}}} \cdot \frac{1}{s} \cancel{e^s} \, ds =$

$$= e^x \int_1^x (-1)\, ds + x e^x \int_1^x \frac{1}{s}\, ds =$$

$$= e^x (-x+1) + x e^x \cdot \log x$$

$$y(x) = c_1 e^x + c_2 x e^x + y_M(x)$$

ΕΞΙΣΩΣΕΙΣ EULER

$$a_n x^n y^{(n)} + a_{n-1} x^{n-1} y^{(n-1)} + \dots + a_1 x y' + a_0 y = 0$$

$a_n \neq 0$, $a_i \in \mathbb{R}$ $(i = 0, 1, \dots, n)$ $\quad x > 0 \vee x < 0$

• $x > 0 \rightarrow$ Θέτω $t = \log x \Leftrightarrow x = e^t$

$$y' = \frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{1}{x} \cdot \frac{dy}{dt} \Rightarrow x y' = \frac{dy}{dt}$$

$$y'' = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dt}\right) = -\frac{1}{x^2}\frac{dy}{dt} + \frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dt}\right) =$$

$$= -\frac{1}{x^2}\frac{dy}{dt} + \frac{1}{x}\frac{d}{dt}\left(\frac{dy}{dt}\right)\frac{dt}{dx} = -\frac{1}{x^2}\frac{dy}{dt} + \frac{1}{x^2}\frac{d^2y}{dt^2} \Rightarrow$$

$$\Rightarrow y'' = \frac{1}{x^2}\left(-\frac{dy}{dt} + \frac{d^2y}{dt^2}\right)$$

Άρα, $x^2 y'' = -\dfrac{dy}{dt} + \dfrac{d^2y}{dt^2}$.

$$y''' = \frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d}{dx}\left(\frac{1}{x^2}\left(-\frac{dy}{dt} + \frac{dy^2}{dt^2}\right)\right) =$$

$$= -\frac{2}{x^3}\left(-\frac{dy}{dt} + \frac{d^2y}{dt^2}\right) + \frac{1}{x^2}\frac{d}{dx}\left(-\frac{dy}{dt} + \frac{d^2y}{dt^2}\right) =$$

$$= -\frac{2}{x^3}\left(-\frac{dy}{dt} + \frac{d^2y}{dt^2}\right) + \frac{1}{x^2}\frac{d}{dt}\left(-\frac{dy}{dt} + \frac{d^2y}{dt^2}\right)\frac{dt}{dx} = \frac{1}{x}$$

$$\Rightarrow x^3 \cdot y''' = 2\frac{dy}{dt} - 3\frac{d^2y}{dt^2} + \frac{d^3y}{dt^3}$$

ΠΑΡΑΔΕΙΓΜΑ 6 (ΣΕΛ 112)

$$x^3 \cdot y''' - x^2 y'' - 2xy' - 4y = 0$$

ΛΥΣΗ

$$2\frac{dy}{dt} - 3\frac{d^2y}{dt^2} + \frac{d^3y}{dt^3} - \left(-\frac{dy}{dt} + \frac{d^2y}{dt^2}\right) - 2\left(\frac{dy}{dt}\right) - 4y = 0$$

$$\Rightarrow \frac{d^3y}{dt^3} - 4\frac{d^2x}{dt^2} + \frac{dy}{dt} - 4y = 0 \Rightarrow$$

$$\Rightarrow \lambda^3 - 4\lambda^2 + \lambda - 4 = 0 \Rightarrow (\lambda - 4)(\lambda^2 + 1) = 0 \Rightarrow$$

$$\Rightarrow \text{ΒΣΛ } \{e^{4t}, \cos t, \sin t\}, \ t > 0$$

$$\overset{t = \log x}{\Rightarrow} \text{ΒΣΛ } \{e^{4\log x}, \cos(\log x), \sin(\log x)\}, \ x > 0$$

ΑΣΚΗΣΗ 6ί (ΣΕΛ 114)

$$x^2 y'' - xy' + y = 0, \quad x > 0 \quad \left| \begin{array}{l} y(1) = 1 \\ y'(1) = 0 \end{array} \right.$$

ΛΥΣΗ

$t = \log x$ και παίρνουμε:

$$-\frac{dy}{dt} + \frac{d^2y}{dt^2} - \frac{dy}{dt} + y = 0, \quad t \in \mathbb{R}$$

$$\frac{d^2y}{dt^2} - \frac{2dy}{dt} + y = 0, \quad t \in \mathbb{R}$$

$\lambda^2 - 2\lambda + 1 = 0 \Rightarrow \lambda = 1$ Διπλή

Άρα ΒΣΛ $\{e^t, te^t\}$

$y(x) = c_1 x + c_2 x \log x, \ x > 0$ ← όλες οι λύσεις

$y(1) = 1 \Rightarrow 1 = c_1 + c_2 \log 1 \Rightarrow c_1 = 1$

$y'(1) = 0 \Rightarrow c_2 = \ldots$

ΑΣΚΗΣΗ 6 iii

$$(x-2)^2 y'' - (x-2)y' + y = 0 \quad , \ x > 2$$

ΛΥΣΗ

Συντομος τροπος

Θετω $u = x-2$, $u>0$

$$y'' = \frac{d^2y}{dx^2} = \ldots = \frac{d^2y}{du^2} \rightarrow u^2y'' - uy' + y = 0$$

Απο την (6 ii) προκυπτει για την ιδια διαφ. εξισωση

$$\bar{y}(u) = c_1 u + c_2 u \log u$$

$$y(x) = \bar{y}(x-2) = c_1(x-2) + c_2(x-2)\log(x-2)$$

Αν δεν την ειχε λυσει θετω $x-2 = \log t$ κλπ...

ΑΣΚΗΣΗ 8 i

$$5x^2y'' - 3xy' + 3y = x^{1/2} \quad , \ x>0$$

ΛΥΣΗ

Θετω $t = \log x \Rightarrow 5\left(-\frac{dy}{dt} + \frac{d^2y}{dt^2}\right) - 3\frac{dy}{dt} + 3y = e^{\frac{1}{2}t}$

$$5\frac{d^2y}{dt^2} - 8\frac{dy}{dt} + 3y = e^{\frac{1}{2}t}$$

χπ... $5\lambda^2 - 8\lambda + 3 = 0$ $\Rightarrow$ $\lambda = 1$, $\lambda = 3/5$

ΒΣΛ $\{e^t, e^{3/5\, t}\}$

ΑΣΚΗΣΗ 9

$$x^2y'' + 2y' + yx = 0 \ , \ y\left(\frac{\pi}{4}\right) = 0 \ , \ y'\left(\frac{\pi}{4}\right) = 1$$

Υποδειξη

Θετω $z = xy \Rightarrow \ldots \Rightarrow z'' + z' = 0 \Rightarrow$

$\Rightarrow$ ΒΣΛ $\left\{\frac{\cos x}{x}, \frac{\sin x}{x}\right\}$.

$$y_0(x) = \frac{\pi}{4\sqrt{2}}\frac{1}{x}(\sin x - \cos x)$$

ΑΣΚΗΣΗ 11

$$x^2y'' + 4xy' + (2+x^2)y = 0$$

ΛΥΣΗ

$$x^2y'' + 4xy' + (2+x^2)y = 0$$

$$x^2y'' + 2x'\cdot y' + 2y + x^2y = 0$$

$$(x^2y)'' + x^2y = 0 \quad \text{Θέτω } z = x^2y$$

$$z'' + z = 0 \Rightarrow \ldots$$